"Theory and implementation of the affine interest rate models: Vasicek and Cox-Ingersoll-Ross"

by

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Chapter 1.

Introduction

This thesis aims to give a better understanding of affine interest rate models and tries to fill the gaps in the general literature of the derivations of the induced bond prices in the Vasicek and the Cox-Ingersoll-Ross short rate models. In the first part that I call mathematical preliminaries concepts such as the Stochastic Integral, Itô’s Lemma, Change of Measure and Stochastic Differential Equations are presentend. The subsequent part, modeling, presents the theoretical framework that is used to model zero coupon bond prices. In this part the concepts of Arbitrage Pricing, Market Price of Risk and the so called Term Structure Partial Differential Equation are described. Then I give a characterization of affine Term Structure Models and calibrate the Vasicek and Cox-Ingersoll-Ross short rate models to 1w Euribor interest rate data.
Chapter 2.

Mathematical Preliminaries

Before I introduce the Vasicek and the Cox-Ingersoll-Ross short rate models, I reiterate the basic assumptions of the Itô Stochastic Integral and the Change of Measure technique. This chapter draws heavily on Mikosch’s book ”Elementary Stochastic Calculus” ([8]).

2.1. Itô Stochastic Integral

In the following a Brownian Motion \((B_t, t \in [0, \infty])\) is a Stochastic Process that is defined on a Probability Space: \((\Omega, \mathcal{F}, \mathbb{P})\) under the following properties:

Definition 1.

- \(B_t\) is a Stochastic Process defined as: \((B_t, t \geq 0) = (B_t(\omega), t \in [0, \infty], \omega \in \Omega)\) on the Outcome Space \(\Omega\).
- It starts at zero: \(B_0 = 0\).
- It has stationary and independent increments.
- For every \(t \geq 0\), \(B_t\) has a normal \(N(0, t)\) distribution.
- It has continuous sample paths.
- It is nowhere differentiable.

It is the theory of Itô calculus that describes integrals of stochastic processes. Generally, the Itô integral on the interval \([0,1]\) for fixed \(\omega \in \Omega\) is written as:

\[
Y_t = \int_0^1 H_t dX_t \quad (2.1)
\]
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In this definition $H_t$ is a locally square integrable process \(^1\) adapted to the filtration generated by a semimartingale $X_t$. This integral resembles the general Riemann-Stieltjes integral on $[0,1]$ $S = \int_0^1 h(t)dg(t)$ for bounded and sufficiently smooth functions $h,g$. Thus it is the purpose of the following paragraphs to investigate why two integral definitions with the same purpose exist.

First, consider the Riemann-Stieltjes Integral and its application in the calculation of the expectation of a random variable:

$$ E(X) = \int_{-\infty}^{\infty} x dF_X(x) $$

It is also known that this integral emerges as the limit of a sequence of Riemann-Stieltjes Sums. For example, for two real valued functions $h$ and $g$ and two sequences of partitions $(\tau_n, \sigma_n)$ of the interval $[0,1]$:

$$ \tau_n : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1 \quad (2.2a) $$

$$ \sigma_n : x_{i-1} \leq y_i \leq x_i \quad \text{for } i = 1, \ldots, n \quad (2.2b) $$

the Riemann-Stieltjes Sums:

$$ S_n(\tau_n, \sigma_n) = S_n = \sum_{i=1}^{n} h(y_i)[g(x_i) - g(x_{i-1})] \quad (2.3) $$

are elements of the sequence $(S_1(\tau_1, \sigma_1), S_2(\tau_2, \sigma_2), \ldots, S_n(\tau_n, \sigma_n))$. If the limiting sum for $n \to \infty$ (the partition becomes ever finer) exists it becomes the Riemann-Stieltjes Integral ($S$). This is written as:

$$ \lim_{n \to \infty} S_n(\tau_n, \sigma_n) = \lim_{n \to \infty} \sum_{i=1}^{n} h(y_i)[g(x_i) - g(x_{i-1})] $$

$$ := \int_0^1 f(x)dg(x) \quad (2.4) $$

$$ = S $$

With this, it is natural to ask whether one can also define, as an integral in the Riemann-Stieltjes sense, the sample path of a Brownian Motion with respect to its increments on the interval $[0,1]$:

\(^1\)The stochastic process $H_t$ is square integrable when $\int_{-\infty}^{\infty} E(H_t^2)dx < \infty$ holds.
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\[ I (B(\omega)) = \int_0^1 B_t(\omega) dB_t(\omega), \ \omega \in \Omega \]

This problem extends to more general functions for the integrand process. In such a case you can replace \( B_t(\omega) \) (in the above expression) with e.g. \( f(t, B_t(\omega)) \), that is a stochastic function of the Brownian motion and time. The answer to this question is "No", in general. For this I review the existence conditions for the Riemann-Stieltjes Integral (The interested reader is referenced to (Mikosch, 1998 [8]) for for a lot more detail).

**Definition 2.** The real valued function \( h \) on \([0, 1]\) is said to have Bounded \( p\)-Variation for some \( p \geq 0 \) if

\[
\sup_\tau \sum_{i=1}^n |h(x_i) - h(x_{i-1})|^p < \infty
\]

with the supremum taken over all possible partitions \( \tau \) on \([0, 1]\).

**Definition 3.** The conditions for the existence of the Riemann-Stieltjes Integral \( \int_0^1 h(x) dg(x) \) are:

- The functions \( h \) and \( g \) do not have discontinuities at the same point \( x \in [0, 1] \)
- The function \( h \) has Bounded \( p\)-Variation and the function \( g \) has Bounded \( q\)-Variation for some \( p \geq 0 \) and \( q \geq 0 \) such that \( p^{-1} + q^{-1} \geq 1 \).

It is known (Taylor,1972) that Brownian Motion sample paths have Bounded \( p\)-Variation for \( p > 2 \) and Undounded \( p\)-Variation for \( p \leq 2 \). From the second condition from Definition 3 it is obvious that the integral \( I(B)(\omega) = \int_0^1 B_t(\omega) dB_t(\omega) \) does not exist as a Riemann-Stieltjes Integral. Therefore, it must be defined in a different sense - this is done with the definition of the Itô Stochastic Integral. As we have seen before, the Riemann-Stieltjes Integral (S) is defined as the limit (for ever finer partitions) of a Riemann-Stieltjes Sum \( (S_n(\tau_n, \sigma_n)) \) (cf. Definition 2.4). Although, it is not possible to define \( I(B)(\omega) = \int_0^1 B_t(\omega) dB_t(\omega) \) as a Riemann-Stieltjes integral \( (S) \), it will turn out to be useful to approach it from a sequence of Riemann-Stieltjes Sums \( (S_n(\tau_{n,1}, \sigma_{n,1})) \) if the partitions are defined in a certain form. Similar to the discussion about the Riemann-Stieltjes Sum \( (\tau_{n,1}, \sigma_{n,1}) \) are defined, although this time on the interval \([0,t]\):

\[
\tau_{n,1}: 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \quad (2.5a)
\]
\[
\sigma_{n,1}: y_i = t_{i-1} \text{ for } i = 1, \ldots, n \quad (2.5b)
\]

Importantly, the intermediate partition \( (\sigma_{n,1}) \) always takes the left value of the interval \([t_{i-1}, t_i]\). A sum in the sequence of Riemann-Stieltjes sums is given by:
Chapter 2. Mathematical Preliminaries

\[ S_n(\tau_{n,1}, \sigma_{n,1}) = \sum_{i=1}^{n} B_{t_i-1} [B_{t_i} - B_{t_i-1}] \]

\[ = \sum_{i=1}^{n} (B_{t_i-1} B_{t_i} - B_{t_i-1}^2) \]

\[ = \sum_{i=1}^{n} (-\frac{1}{2}(B_{t_i} - B_{t_i-1})^2 + \frac{1}{2}B_{t_i}^2 + \frac{1}{2}B_{t_i-1}^2 - B_{t_i-1}^2) \]

\[ = \frac{1}{2} \sum_{i=1}^{n} (B_{t_i}^2 - B_{t_i-1}^2) - \frac{1}{2} \sum_{i=1}^{n} (B_{t_i} - B_{t_i-1})^2 \]

\[ = \frac{1}{2} B_{t_n}^2 - B_{t_0}^2 - \frac{1}{2} \sum_{i=1}^{n} (B_{t_i} - B_{t_i-1})^2 \]

\[ = \frac{1}{2} \left[ B_{t_n}^2 - \sum_{i=1}^{n} (B_{t_i} - B_{t_i-1})^2 \right] \quad (2.6) \]

Here the third equality follows from adding and subtracting \( \frac{1}{2}B_{t_i} \) and \( \frac{1}{2}B_{t_i-1} \) and writing it in terms of the binomial formula. Equality five follows from writing out the “Telescope Sum” \( \frac{1}{2}(B_{t_1}^2 - B_{t_0}^2 + B_{t_2}^2 - B_{t_1}^2 + \cdots + B_{t_n}^2) \). Next the moments of the sum \( Q_n(t) = \sum_{i=1}^{n} (B_{t_i} - B_{t_i-1})^2 \) (it is a sequence of Random Variables for \( n \)) are inspected:

\[ E(Q_n(t)) = \sum_{i=1}^{n} \left[ E(B_{t_i}^2) - 2E(B_{t_i}B_{t_i-1}) + E(B_{t_i-1}^2) \right] \]

\[ = \sum_{i=1}^{n} (t_i - 2t_{i-1} + t_{i-1}) \]

\[ = t \quad (2.7) \]

Here the second equality follows from the known fact: \( \text{Cov}(B_t, B_s) = E(B_tB_s) = t \wedge s \). Also note that \( E(Q_n(t)) \) is only a function of \( t \) and does not depend on the number of partitions \( n \). To calculate the Variance of \( E(Q_n(t)) \), it is useful to remember the independent increments property as given in Definition 1 and the fact that an increment can be written as \( B_{t_i} - B_{t_i-1} = B_{t_i-t_{i-1}} \):
\[
\text{Var}(Q_n(t)) = E \left[ \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \sum_{j=1}^{n} (B_{t_j} - B_{t_{j-1}})^2 \right] - \left[ \sum_{i=1}^{n} E[(B_{t_i} - B_{t_{i-1}})^2] \right]^2
\]

\[
= E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} B_{t_i-t_{i-1}}^2 B_{t_j-t_{j-1}}^2 \right] - \sum_{i=1}^{n} \sum_{j=1}^{n} E(B_{t_i-t_{i-1}}^2)E(B_{t_j-t_{j-1}}^2)
\]

\[
= \sum_{i=1}^{n} E((t_i - t_{i-1})^2) - \sum_{i=1}^{n} (t_i - t_{i-1})^2 + 0
\]

\[
= \sum_{i=1}^{n} 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2
\]

\[
= 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2
\]

The fourth equality follows from rewriting: \( B_{t_i-t_{i-1}}^4 = ((t_i - t_{i-1})^2 B_i)^4 \) and using the independence property of the increments \( E(B_{t_i-t_{i-1}} B_{t_j-t_{j-1}}) = E(B_{t_i-t_{i-1}})E(B_{t_j-t_{j-1}}) \). The fifth equality follows from the properties of a standard normal variable \( E(B_i^2) = 3, \ for \ B_i \sim N(0, 1) \). From the definition of the partition \( \tau_{n,1} \) (cf. definition 2.5a) one can see that as \( n \to \infty \) then \( t_i - t_{i-1} \to 0 \) (the intervals will become ever shorter) which implies:

\[
\lim_{n \to \infty} 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 = \lim_{n \to \infty} \text{Var}(Q_n(t)) = 0
\]

Since \( \text{Var}(Q_n(t)) = E(Q_n(t) - t)^2 \to 0 \) most of the work for the proof for convergence in Mean Square is done. The conditions for convergence in Mean Square can be found in many statistics books, as for example in Mikosch,1998 p.187 [8] ). Now one can draw the conclusion that the integral \( I_f(B)(\omega) = \int_0^t B_s(\omega)dB_s(\omega) \) does not converge in the Riemann-Stieltjes sense, but that it converges in a Mean Squared sense. The value of the integral \( I(B)(\omega) \) is usually denoted as the value of the Itô Stochastic Integral:
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\[ I_t(B) = \int_0^t B_s dB_s \]  
\[ = \frac{1}{2} \left[ B_t^2 - \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \right] \]  
\[ = \frac{1}{2}(B_t^2 - t) \]

From the discussion above it is important to note that the Integrand \((B_s)\) in the Integral 2.9 is a Brownian Motion evaluated at the left end point on the interval \([t_{i-1}, t_i]\) (That is the reason why it can be represented as a Riemann-Stieltjes Sum). In order to define the Itô Stochastic Integral for more general Integrand processes\((H)\) certain assumptions on \(H = (H_t, t \in [0, T])\) are required:

**Assumptions 1.** For (stochastic) process \(H = (H_t, t \in [0, T])\)

(a) \(H\) is adapted to Brownian Motion.

(b) The integral \(\int_0^T E(H_s^2)\) is finite.

Assumption 1.1 states that the Integrand Process must be a function of past and present Brownian motion values. The usual notation is to describe this property in terms of \(Sigma-Fields\). For a Brownian Motion \(B = (B_t, t \geq 0)\) the \(Sigma-Field\) that includes its past and present values is denoted by:

\[ \mathcal{F}_t = \sigma(B_s, s \leq t), \ 0 \leq t. \]

\(\mathcal{F}_t\) is called the Natural Filtration of \(B\). In fact, \(\mathcal{F}_t\) is the collection of all possible subsets of \(B(\omega)\) on the Outcome Space \(\Omega\) that can be constructed using the binary operators: \(\cap, \cup, \neg, \emptyset\) (“Intersection”, “Union”, “Complement”, “Emptyset”). In this context the following definition of adaptedness proves to be useful:

**Definition 4.** A stochastic process \((H_t, t \geq 0)\) is adapted to the filtration \((\mathcal{F}_t, t \geq 0)\) if:

\[ \sigma(H_t) \subset \mathcal{F}_t, \ \text{for all } t \geq 0 \]

With assumptions 1 the Itô Stochastic Integral is defined by:

**Definition 5.** The mean square limit \(I_t(H)\) is called the Itô Stochastic Integral of \(H\). It is written as

\[ I_t(H) = \int_0^t H_s dB_s, \ t \in [0, T] \]

The Itô Stochastic Integral has the following properties:
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- It is a martingale with respect to \((\mathcal{F}_t, t \in [0, T])\) (The Natural Brownian Filtration).
- It has expectation zero.
- It satisfies the Isometry Property:
  \[
  E \left[ \left( \int_0^t H_s dB_s \right)^2 \right] = \int_0^t E(H_s^2) ds, \quad t \in [0, T].
  \]
- It has continuous sample paths.

The purpose of this section was to define the Itô Stochastic Integral. It is important, because it is the defining property of Itô Stochastic Differential Equations that play an important role in financial modeling. In the next section, I will consider Itô Stochastic Differential Equations.

### 2.2. Itô Stochastic Differential Equations

In this section I will briefly introduce Itô Stochastic Differential Equations, the Itô Formula and provide the solution processes for the two most common Itô Stochastic Differential Equations:

a The Ornstein-Uhlenbeck Process.

b The Geometric Brownian Motion.

#### 2.2.1. Strong and Weak Solutions

A general description for a Itô Stochastic Differential Equation, for a given Stochastic Process \((X_t, t \in [0, T])\) can be given by:

\[
\begin{align*}
  dX_t &= a(t, X_t) dt + b(t, X_t) dB_t \\
  X_0(\omega) &= Y(\omega)
\end{align*}
\]  

(2.12) \hspace{1cm} (2.13)

Here the initial condition \((X_0)\) as given in equality 2.13 is a random variable, although in most applications it is set to be constant. More importantly, equality 2.12 tells us that a (very) small change in \(X_t\) \((dX_t)\) comes from a (very) small change in time \((dt)\), scaled by factor \(a(t, X_t)\), and from a (very) small change in in the Brownian Motion \((dB_t)\), scaled by factor \(b(t, X_t)\). In light of the last property in Definition 1 we cannot interpret equality 2.12 in terms of derivatives (Unlike Ordinary Differential Equations). Thus there is no gain by writing equality 2.12 as:
\[
\frac{dX_t}{dt} = a(t, X_t) + b(t, X_t) \frac{dB_t}{dt}
\]

It is more useful to consider the equivalent integral representation of equality 2.12:

\[
X_t = X_0(\omega) + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s, \quad 0 \leq t \leq T
\]  

(2.14)

With this representation the first integral of equality 2.14 is a Riemann Integral and the second integral is the Itô Stochastic Integral. Despite of being given in integral form equality 2.14 is called the Itô Stochastic Differential Equation (SDE) or an Itô Process. In case there exists a closed form solution to equality 2.14 there are two forms for the solution to consider (The so called strong and weak -solutions). Strong Solutions are based on the sample path of the underlying Brownian Motion. Although the functional form of the solution would stay intact, one would obtain a different solution for equality 2.14 in a different state of the world. A definition of a strong form solution is given in the following Definition:

Definition 6. A strong solution to the Itô Stochastic Differential Equation is a stochastic process \(X = (X_t, t \in [0, T])\) that:

- Is adapted to Brownian Motion.
- Has well defined Integrals in its Itô Stochastic Differential Equation form.
- Is a function of the underlying Brownian Motion sample path and of the coefficient functions \(a(t, x)\) and \(b(t, x)\).

Next I give the sufficient existence conditions for the Strong Solution under the condition that the initial condition is fairly regular: \(E(X_0(\omega)^2) < \infty\) (These results are among others proved in Kloeden and Platen (1992))

Definition 7. The Itô Stochastic Differential Equation will have a unique strong solution \(X\) on \([0, T]\) if:

for \(t\) in \([0, T]\); \(x, y \in \mathbb{R}\) and \(K \geq 0\)

- \(a(t, x)\) and \(b(t, x)\) are continuous.
- \(|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|\)

The Strong Solution is useful in financial modeling, because with the help of computers one can simulate multiple sample paths of the underlying process. For a Weak Solution the sample paths of \(a(t, x), b(t, x)\) and \((B_t, t \in [0, T])\) are not important. A Weak Solution \(X = (X_t, t \in [0, T])\) finds a Brownian Motion that satisfies the Itô Stochastic Differential Equation while \(X_0, a(t, x)\) and \(b(t, x)\) are fixed. A Weak Solution is sufficient
to determine distributional properties of X, such as its expectation, variance or covariance. In general it is difficult to solve Itô Stochastic Differential Equations. One way to solve SDEs is to guess a solution and then use the Itô Formula (This is a chain rule for Processes that contain the Itô Integral) to check whether the guessed solution satisfies the underlying SDE. Before I give the solutions to the two processes mentioned at the beginning of this section it will be useful to review the definition of the Itô Formula.

### 2.2.2. Itô Formula

The Itô Formula can be interpreted as the chain rule of derivation for the Itô Stochastic Integral. Recall from the previous section the definition and the value of the Itô Stochastic Integral (I repeat them here for convenience)

\[
I_t(B) = \int_0^t B_s dB_s
\]

\[
= \frac{1}{2}(B_t^2 - t)
\]

(2.15)

(2.16)

It is useful to compare the Itô Stochastic Integral to its Riemann-Stieltjes analogue.

\[
f(b) = \int_0^t b(s) db(s)
\]

(2.17)

The solution to the integral 2.17 can be found by application of the Chain Rule: \([f(b(t))]' = f'(b(t))b'(t)\) and assuming \(b(0) = 0\)

\[
f(b) = \int_0^t b(s) db(s)
\]

\[
= \frac{1}{2}b^2|_0^t
\]

\[
= \frac{1}{2}b^2
\]

(2.18)

Comparing the value of equality 2.16 to 2.18 one observes that the value of the Itô Stochastic Integral corrects the value of the Riemann-Stieltjes Integral downwards by \(-\frac{1}{2}t\). Thus the chain rule does not hold for Itô Stochastic Integrals.

Moreover, the result of the Chain Rule \([f(b(t))]'\) is equal to the first term of the Taylor expansion for \(f(b(t))\) under certain conditions on \(b(t)\). For this one rewrites \([f(b(t))]'\) as:

\[
10
\]
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\[
[f(b(t))]' = \frac{df(b(t))}{dt} = f'(b(t))db(t) \quad (2.19)
\]

And assuming in the Taylor expansion for \( f(b(t)) \) that \( b(t) \) is smooth enough to neglect higher order terms:

\[
df(b(t)) = f'(b(t))db(t) + \frac{1}{2}f''(b(t))(db(t))^2 + \ldots \quad (2.20)
\]

\[
= f'(b(t))db(t) \quad (2.21)
\]

But since Brownian Motions are nowhere differentiable it makes intuitive sense that more terms should be included in the Taylor expansion around \( df(B_t) \). First one can replace \( b(t) \) with \( B_t \) in the Taylor expansion 2.20 to obtain:

\[
df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \ldots \quad (2.22)
\]

From the discussion in the previous section (cf. Riemann-Stieltjes Sum Representation 2.6) we can write (For given partitions):

\[
(dB_t)^2 = (B_{t_i} - B_{t_{i-1}})^2 \quad (2.23)
\]

Moreover we have seen that approximation 2.23 converges in Mean Square:

\[
(B_{t_i} - B_{t_{i-1}})^2 \xrightarrow{m.s} t_i - t_{i-1} \quad (2.24)
\]

\[
\lim_{n \to \infty} t_i - t_{i-1} = dt \quad (2.25)
\]

With this insight the Taylor expansion in 2.22 is given by:

\[
df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \ldots
= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \quad (2.26)
\]

In final expression 2.26 all higher order terms are neglected. The formal definition of the simple version of the Itô Formula is obtained by integrating expression 2.26 on the interval \([s, t]\):
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**Definition 8.** For a twice continuously differential function $f(B)$

$$
\int_s^t df(B_u) := f(B_t) - f(B_s) = \int_s^t f'(B_u)dB_u + \frac{1}{2} \int_s^t f''(B_u)du, \quad s < t
$$

is the simple Itô Formula.

A useful extension of the Itô Formula is when it is defined for functions of time and an Itô Process: $f(t, X_t)$. Here $X_t$ is given similar to expression 2.14:

$$
X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, \quad s < t \tag{2.27}
$$

and

$$
dX_t = a(t, X_t)dt + b(t, X_t)dB_t \tag{2.28}
$$

In this case the Itô Formula is given by:

**Definition 9.** For an Itô Process $X_t$ and a function $f(t, x)$ with continuous second order partial derivatives

$$
\int_s^t df(u, X_u) := f(t, X_t) - f(s, X_s) = \int_s^t \left[ f_1(u, X_u) + \frac{1}{2} [b(u, X_u)]^2 f_{22}(u, X_u) \right] du + \int_s^t f_2(u, X_u)dX_u, \quad s < t
$$

and

$$
dX_u = a(u, X_u)dt + b(u, X_u)dB_u
$$

is the extended Itô Formula for a function of time and an Itô Process.

In the extended Itô Formula $f_1(\cdot, \cdot); f_2(\cdot, \cdot); f_{22}(\cdot, \cdot)$ are the partial derivatives of the first and second arguments respectively:

$$
\begin{align*}
f_1(t, x) &= \frac{\partial f(t, x)}{\partial t} \\
f_2(t, x) &= \frac{\partial f(t, x)}{\partial x} \\
f_{22}(t, x) &= \frac{\partial^2 f(t, x)}{\partial x^2}
\end{align*}
$$

The final extension of the Itô Formula is given for functions of multiple Itô processes:
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**Definition 10.** For k Itô Processes $X^{(1)}_t, \ldots, X^{(k)}_t$ and a function $f(t,x_1,\ldots,x_k)$ with continuous second order partial derivatives for $s < t$

$$
\int_s^t df(u, X^{(1)}_u, \ldots, X^{(k)}_u) := f(t, X^{(1)}_t, \ldots, X^{(k)}_t) - f(s, X^{(1)}_s, \ldots, X^{(k)}_s)
$$

$$
= \int_s^t f_1(u, X^{(1)}_u, \ldots, X^{(k)}_u)du + \sum_{i=2}^{k+1} \int_s^t f_i(u, X^{(1)}_u, \ldots, X^{(k)}_u)dX^{(i-1)}_u
$$

$$
+ \frac{1}{2} \sum_{i=2}^{k+1} \sum_{j=2}^{k+1} \int_s^t f_{i \ldots j}(u, X^{(1)}_u, \ldots, X^{(k)}_u)b^{(i-1)}(u, X^{(i-1)}_u) \ldots b^{(j-1)}(u, X^{(j-1)}_u)du
$$

and

$$
dX^{(i)}_u = a^{(i)}(u, X^{(i)}_u)dt + b^{(i)}(u, X^{(i)}_u)dB_u, \quad i \in \{2, \ldots, k\}
$$

is the Multi Itô Formula for a function of time and k Itô Processes. Here $f_i(t,x_1,\ldots,x_k), f_{i \ldots j}(t,x_1,\ldots,x_k)$ are the partial derivatives of $f(t,x_1,\ldots,x_k)$ with respect to the $i$th and the $i$th to $j$th variable.

Recall from the previous discussion that the Itô formula can be used to find Solutions to Itô Stochastic Differential Equations. In the next section I will provide solutions to the Geometric Brownian Motion and the Ornstein-Uhlenbeck Processes.

### 2.2.3. The Linear Differential Equation

The General Linear Differential Equation is given by:

$$
X_t = X_0 + \int_0^t [c_1(s)X_s + c_2(s)]ds + \int_0^t [\sigma_1(s)X_s + \sigma_2(s)]dB_s, \quad t \in [0,T]
$$

(2.29)

Since the coefficient functions in equality 2.29: $c_1(t), c_2(t), \sigma_1(t), \sigma_2(t)$ are deterministic and continuous one can see from Definition 7 that the General Linear Differential Equation has a unique Strong Solution. The two most widely applied Linear SDEs of this class are:

a. The Linear Equation with Additive Noise.

b. The Linear Equation with Multiplicative Noise.

The **Linear Equation with Additive Noise** is obtained from expression 2.29 by setting $\sigma_1(t) = 0$. It is given as:
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\[ X_t = X_0 + \int_0^t [c_1(s)X_s + c_2(s)]ds + \int_0^t \sigma_2(s)dB_s, \quad t \in [0, T] \]  
\[ \text{and} \]
\[ dX_t = [c_1(t)X_t + c_2(t)]dt + \sigma_2(t)dB_t \]

(2.30)

One way to find the solution for expression 2.30 is by writing:

\[ X_t = Y_t \exp \left( \int_0^t c_1(s)ds \right) \]
\[ \iff \]
\[ Y_t = X_t \exp \left( -\int_0^t c_1(s)ds \right) \]
\[ = f(t, X_t) \]

(2.32)

(2.33)

(2.34)

Assuming that the function in 2.34 is smooth enough one can apply the Itô Formula from Definition 9 to it. The Partial Derivatives are:

\[ f_1(t, x) = -c_1(t)x \exp \left( -\int_0^t c_1(s)ds \right) \]
\[ f_2(t, x) = \exp \left( -\int_0^t c_1(s)ds \right) \]
\[ f_{22}(t, x) = 0 \]

Plugging these into the Itô Formula (Definition 9) yields:
\[
\int_0^t df(t, X_t) = f(t, X_t) - f(0, X_0)
\]
\[
= \int_0^t -c_1(s)X_s \exp \left( - \int_0^s c_1(u)du \right) ds
+ \int_0^t \exp \left( - \int_0^s c_1(u)du \right) \left[ \{c_1(s)X_s + c_2(s)\} ds + \sigma_2(s)dB_s \right]
\]
\[
= \int_0^t \left[ -c_1(s)X_s \exp \left( - \int_0^s c_1(u)du \right) + c_1(s)X_s \exp \left( - \int_0^s c_1(u)du \right) \right] ds
+ \int_0^t \exp \left( - \int_0^s c_1(u)du \right) c_2(s)ds + \int_0^t \exp \left( - \int_0^s c_1(u)du \right) \sigma_2(s)dB_s
\]
\[
= \int_0^t \exp \left( - \int_0^s c_1(u)du \right) c_2(s)ds + \int_0^t \exp \left( - \int_0^s c_1(u)du \right) \sigma_2(s)dB_s
\]

(2.35)

From expressions 2.33 and 2.31 it is clear that \( X_0 = f(0, X_0) \). Therefore (replacing \( f(t, X_t) \) by its expression in 2.33) one can write:

\[ X_t = y(t)^{-1} \left( X_0 + \int_0^t y(s)c_2(s)ds + \int_0^t y(s)\sigma_2(s)dB_s \right) \]  

(2.36)

and

\[ y(t) = \exp \left( - \int_0^t c_1(s)ds \right) \]  

(2.37)

As an Example, the Ornstein-Uhlenbeck (OU) Process belongs to the class of Linear Equations with Additive Noise. Its Stochastic Differential Equation is obtained by setting \( c_1(t) = c \), \( c_2(t) = 0 \) and \( \sigma_2(t) = \sigma \) in expression 2.30. With the generic Solution for Linear Stochastic Processes as given in 2.36 it is easy to find the solution for the Ornstein-Uhlenbeck Process. Its SDE and Solution, respectively:

\[ X_t = X_0 + \int_0^t cX_s ds + \int_0^t \sigma dB_s \]  

(2.38)

\[ X_t = \exp(ct)X_0 + \exp(ct) \int_0^t \sigma \exp(-cs)dB_s \]  

(2.39)

The second special case of the class of Linear Stochastic Differential Equations is the so called Linear Stochastic Equation With Multiplicative Noise (It is also called the
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Homogeneous Linear Equation. It is obtained from the general form 2.29 by setting \( c_2(t) = \sigma_2(t) = 0 \) and given by:

\[
X_t = X_0 + \int_0^t c_1(s)X_sds + \int_0^t \sigma_1(s)X_sdB_s, \quad t \in [0, T] \tag{2.40}
\]

and
\[
dX_t = c_1(t)X_tdt + \sigma_1(t)X_tdB_t \tag{2.41}
\]

The form of the solution to equation 2.40 can be guessed from its expression in differentials (Expression 2.41). From this a solution involving the exponential function is plausible, because both the Riemann Integral and the Itô Stochastic Integral include the current value of the process. This implies that if \( X_0 > 0 \) also \( X_t > 0 \). One approach to find the solution for \( X_t \) is to find a solution to a transformed process \( Y_t \) and then to transform \( Y_t \) back to \( X_t \).

For the Linear Multiplicative Noise Solution one defines:

\[
Y_t := \ln\left(\frac{X_t}{X_0}\right) \tag{2.42}
\]

\[
= f(X_t) \tag{2.43}
\]

With this definition \( f(X_0) = 0 \). The Partial Derivatives for \( f(x) \) are:

\[
f_1(t, x) = 0
\]

\[
f_2(t, x) = x^{-1}
\]

\[
f_{22}(t, x) = -x^{-2}
\]

Applying the extended Itô Formula (cf. Definition 9) using these Partial Derivatives and the Differential Form in 2.41 with \( a(t, X_t) = c_1(t)X_t \) and \( b(t, X_t) = \sigma_1(t)X_t \):

\[
\int_0^t df(X_t) = f(X_t) - f(X_0)
\]

\[
= \int_0^t -\frac{1}{2} \sigma(s)^2 + \int_0^t X_s^{-1}(c_1(s)X_sds + \sigma_1(s)X_sdB_s)
\]

\[
= \int_0^t (c_1(s) - \frac{1}{2} \sigma(s)^2)ds + \int_0^t \sigma(s)dB_s \tag{2.44}
\]
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Inserting into the Solution 2.44 the initial Definition \( \ln(X_t/X_0) = f(X_t) \) (cf. 2.42) and relying on \( f(X_0) = 0 \) one sees that the Solution to \( X_t \) is:

\[
X_t = X_0 \exp\left(\int_0^t [c_1(s) - \frac{1}{2}\sigma_1(s)^2] ds + \int_0^t \sigma_1(s) dB_s\right)
\]

(2.45)

A special case of the Linear Stochastic Differential Equation is the so called Geometric Brownian Motion (GBM). This Linear Stochastic Differential Equation is obtained from the generic one in 2.29 by setting \( c_2(t) = \sigma_2(t) = 0, c_1(t) = c \) and \( \sigma_1(t) = \sigma \). The SDE and Solution of the Geometric Brownian Motion is then:

\[
X_t = X_0 + \int_0^t cX_s dB_s + \int_0^t \sigma X_s dB_s
\]

(2.46)

\[
X_t = X_0 \exp\left([c - \frac{1}{2}\sigma^2]t + \sigma B_t\right)
\]

(2.47)

In this section Itô Stochastic Differential Equations and their Solutions were introduced. Particularly, the two processes that were introduced towards the end of the section (The Ornstein-Uhlenbeck and the Geometric Brownian Motion) will become important in the coming sections, because they are often used in financial modeling. In the subsequent section I will introduce the concept of the Equivalent Martingale Measure that is an important concept in financial modeling.

2.2.4. Change of Measure

The main purpose of the Change of Measure technique is to eliminate the drift term in a Itô Stochastic Differential Equation so that its Solution Process becomes a martingale (Recall from Property 1 of Definition 5 that the Itô Stochastic Integral is a martingale). In financial modeling the Martingale Property is applied by pricing derivative securities as conditional expectations of a payoff function conditional on some Sigma-Field. The results presented so far assume that the Stochastic Process \((B_t, t \geq 0)\) (cf. Definition 5 for all its properties) is defined on the Probability Space: \((\Omega, \mathcal{F}, P)\). Here \( \Omega \) defines the Outcome Space, \( \mathcal{F} \) is the Sigma-Field of \((B_t, t \geq 0)\) on \( \Omega \), and \( P \) is the probability measure that assigns values to the Outcomes \( \omega \in \Omega \). Under these assumptions (for constant \( q \neq 0 \)) processes of the form:

\[
\tilde{B}_t = B_t + qt, \quad t \in [0, T]
\]

(2.48)

are not Standard Brownian Motion under \( P \) (e.g: \( E(\tilde{B}_t) \neq 0 \)). However changing the measure from \( P \) to, say, \( Q \) the Process \( \tilde{B}_t \) can be transformed into a Standard Brownian Motion. Before presenting the conditions that are required to transform processes of
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the form \( \tilde{B}_t \) into Standard Brownian Motion, I introduce the Definition of the Radon-Nikodym Theorem. It is this Theorem that allows a formal Change of Measure:

**Definition 11.** If for two probability measures \( P, Q \) on the Probability Space: \((\Omega, \mathcal{F})\) there exists a non-negative function \( f \) such that

\[
Q(A) = \int_A f(\omega) dP(\omega), \quad A \in \mathcal{F}
\]

then \( f \) is called the density of \( Q \) with respect to \( P \), and \( Q \) is said to be absolutely continuous with respect to \( P \) \((Q \ll P)\). Moreover \( P \) and \( Q \) are said to be equivalent probability measures if \( Q \ll P \) and \( P \ll Q \).

The conditions that ensure that processes in the form of equality 2.48 become Standard Brownian Motion are summarized under Girsanov’s Theorem. A proof of Girsanov’s Theorem is given in Karatzas and Shreve (1988):

**Definition 12.** Girsanov’s Theorem states that an Equivalent Martingale Measure \( Q \) is defined because:

- **The Stochastic Process**

  \[
  M_t = \exp \left( -qB_t - \frac{1}{2}q^2 t \right), \quad t \in [0, T]
  \]

  is a martingale under \( P \) with respect to \( \mathcal{F}_t = \sigma(B_s, s \leq t), t \in [0, T] \) (The Natural Brownian Filtration).

- **\( Q(A) = \int_A M_T(\omega) dP(\omega) \), \( A \in \mathcal{F} \)**

  defines a Probability Measure \( Q \) on \( \mathcal{F} \) which is equivalent to \( P \).

- **Under \( Q \) the process**

  \[
  \tilde{B}_t = B_t + qt, \quad t \in [0, T]
  \]

  is a Standard Brownian Motion.

- **The process \( \tilde{B}_t \) is adapted to \( \mathcal{F}_t \) (The Natural Brownian Filtration)**

It is constructive to see how the second and third Properties in Definition 12 work for the Normal Distribution. For a process \( \tilde{B}_t \) and \( \mathcal{F}_t = \sigma(B_s, s \leq t), t \in [0, T] \) (With \((B_t, t \geq 0 \) under \( P \) a N(0,t) random variable) :

\[
\tilde{B}_t = B_t + qt, \quad t \in [0, T] \tag{2.49}
\]

\[
d\tilde{B}_t = dB_t + qdt \tag{2.50}
\]
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Under the Measure $\mathbb{P}$:

$$
\mathbb{P}(x) = \int_{-\infty}^{x} d\mathbb{P}^t \omega, \quad x \in \mathcal{F}_t
$$

$$
= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2} \frac{[\tilde{B}_t - qt]^2}{t} \right) d\tilde{B}_t \quad \text{with}
$$

$$
E_{\mathbb{P}}(\tilde{B}_t) = qt \quad (2.51)
$$

$$
V_{\mathbb{P}}(\tilde{B}_t) = t \quad (2.52)
$$

Under the Measure $\mathbb{Q}$:

$$
\mathbb{Q}(x) = \int_{-\infty}^{x} M_t(\omega) d\mathbb{P}(\omega), \quad x \in \mathcal{F}_t
$$

$$
= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp \left( -B_t - \frac{1}{2} q^2 t \right) \exp \left( -\frac{1}{2} \frac{[\tilde{B}_t - qt]^2}{t} \right) d\tilde{B}_t
$$

$$
= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2} \frac{B_t^2}{t} - B_t - \frac{1}{2} q^2 t \right) d\tilde{B}_t
$$

$$
= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2} \frac{[B_t^2 + 2qB_t t + q^2 t^2]}{t} \right) d\tilde{B}_t
$$

$$
= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2} \frac{[B_t + qt]^2}{t} \right) d\tilde{B}_t \quad \text{with}
$$

$$
E_{\mathbb{Q}}(\tilde{B}_t) = 0 \quad (2.53)
$$

$$
V_{\mathbb{Q}}(\tilde{B}_t) = t \quad (2.54)
$$

Because changing the Measure from $\mathbb{P}$ to $\mathbb{Q}$ eliminates the drift term under the new Measure $\mathbb{Q}$, this Measure is called the Equivalent Martingale Measure. Because the Change of Measure technique is useful in pricing derivatives many models are set up under the Measure $\mathbb{Q}$ which is also called the Risk-Neutral Measure.
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Modeling

3.1. Short Rate

In order to set up a modeling framework to price interest rate derivatives, the modeler can either assume that it is constant or that it can be represented as a stochastic process. As an Example consider the 1 week Euribor interest rate:

Figure 3.1.: 1 week Euribor 01/2004 - 01/2006
I assume that the interest rate plotted in the figure above represents the short rate. Specifically, I have decided to model the short rate process using two models:

- Vasicek Model
- Cox-Ingersoll-Ross Model

I use the Vasicek Model, because of its relative simplicity and its appealing properties (that will be derived in the coming text). I, in addition, use the Cox-Ingersoll-Ross model, because it gives more realistic results than Vasicek and it is (in more fine tuned forms) widely used in practice (Yi, 2005 [12]). Before the interest rate models are introduced it is necessary to determine the conditions under which they are used in applications. It will be important to specify the so called Market Price Of Risk process for which it is constructive to know about the Arbitrage Pricing Theory.

### 3.1.1. Arbitrage Pricing

Arbitrage Pricing Theory is the approach that uses replicating portfolios to price derivatives. It is assumed that the economy consists of \( d + 1 \) non-dividend paying traded securities that are modeled as Semi-Martingales:

\[
s_t = (S_t^0 = C_t, S_t^1, \ldots, S_t^d), \quad t \in [0, T]
\]

Here the Process \((C_t, t \in [0, T])\) is the so called Cash Account that satisfies the Deterministic Differential Equation:

\[
dC_t = r_t C_t dt \tag{3.1}
\]

\[
C_0 = 1 \tag{3.2}
\]

with Solution

\[
C_t = C_0 \exp \left( \int_0^t r_s ds \right) \tag{3.3}
\]

The process \(r_t\) is the so called Short-Rate that models the infinitesimal (risk-free) interest rate for the interval \(dt\). An Admissible Trading Strategy is given by a predictable Stochastic Process:

\[
\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d), \quad t \in [0, T]
\]

The Wealth Process of a given Trading Strategy \(\delta_t\) is:
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\[ X_t^\delta = \delta_t s_t, \quad t \in [0, T] \]
\[ = \delta_0^C t + \delta_1^1 S_1 t + \cdots + \delta_d^d S_d t \]

The Trading Strategy \( \delta_t \) is called \textit{Self-Financing} if the Wealth Process satisfies:

\[ dX_t^\delta = \delta_t' ds_t \quad (3.4) \]

Then, for maturity date \( T \), the \( \mathcal{F}_T \) measureable random variable \( H_T \) is \textit{Replicable} if there exists a Trading Strategy \( \delta_t \) such that:

\[ X_T^\delta = H_T \]

If the Random Variable \( H_t \) derives its value from a traded security it is called a Contingent Claim. By the Law of One Price the price of a Replicable Contingent Claim for \( t \leq T \) is:

\[ \pi^H_t = X_t^\delta \quad (3.5) \]

If all the Contingent Claims in an economy are replicable the economy is said to be \textit{Complete}.

Using the definitions above two important Theorems are specified under the Arbitrage Pricing Theory:

\textbf{Definition 13.} The First Fundamental Theorem of Arbitrage Pricing states that the economy is free from arbitrage opportunities if and only if there exists an Equivalent Martingale Measure \( \mathbb{Q} \) with respect to which the Discounted Asset Prices:

\[ \frac{S_t^k}{C_t}, \text{ for } k = \{1, 2, \ldots, d\} \]

are martingales.

The Second Fundamental Theorem of Arbitrage Pricing states that for complete markets the Equivalent Martingale Measure \( \mathbb{Q} \) is unique.

The next section introduces the Market-Price of Risk Process which is important for modeling interest rates.
3.1.2. Market Price of Risk

The following part derives from Chapter 3 in Hurd & Grasselli (2010, [5]). From the previous section we know that a Contingent Claim $H_t$ can be priced if there exists a Trading Strategy $\delta_t$ in the traded securities $s$ that replicates it. The price of non-defaultable zero coupon bond is an example of a contingent claim that cannot be replicated with $\delta_t$ and $s$ alone. Its price is given by:

$$P_t(T) = \begin{cases} 1 & \text{if } t = T \\ p^T(t, r_t) & \text{if } t < T \end{cases}$$ (3.6)

Thus the bond price is specified as a function of two arguments $t$ and the Short Rate $r_t$ whereas the maturity $T \in [0, \infty)$ is a parameter. The Stochastic Differential Equation of the Short Rate is given as:

$$r_t = r_0 + \int_0^t a(t, r_t)dt + \int_0^t b(t, r_t)dB_t$$ (3.7)

This set-up is called a One Factor Short Rate Model. Because the bond price inherits its dynamics from the Short Rate and the Short Rate dynamics are driven by only one source (Brownian Motion), hence the name. Using the Itô Formula (cf. Definition 9 on page 12) on the function $p^T(t, r_t)$ and surpressing the arguments in $a(t, r)$ and $b(t, r)$ one obtains the Bond Partial Differential Equation:

$$p^T(t, r_t) = p^T(0, r_0) + \int_0^t [p^T_1(s, r_s) + \frac{1}{2} b^2 p^T_{22}(s, r_s) + a p^T_2(s, r_s)]ds + \int_0^t b p^T_2(s, r_s)dB_s$$ (3.8)

Equivalently, it is assumed that the bond price can be represented by a Generalized Linear Equation with Multiplicative Noise and stochastic coefficients (cf. Equation 2.40 on page 16)

Assumptions 2. Bond prices $p^S(t, r_t)$ for maturities $S \in [0, T]$ are determined through:

$$p^S(t, r_t) = p^S(0, r_0) + \int_0^t M^S(s, r_s)p^S(s, r_s)ds + \int_0^t \Sigma^S(s, r_s)p^S(s, r_s)dB_s, \quad t \in [0, T]$$

with

$$M^S(t, r_t)p^S(t, r_t) = p^S_1(t, r_t) + \frac{1}{2} b^2 p^S_{22}(t, r_t) + a p^S_2(t, r_t)$$ (3.9)

$$\Sigma^S(t, r_t)p^S(t, r_t) = b p^S_2(t, r_t)$$ (3.10)
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Now consider a Self-Financing portfolio of two bonds with maturities $T$ and $S < T$ and Trading Strategies $\delta^T_t$ and $\delta^S_t$, respectively. The Wealth of the portfolio is given by:

$$X^\delta_t = \delta^T_t p^T(t, r_t) + \delta^S_t p^S(t, r_t)$$  \hspace{1cm} (3.11)

From the Self-Financing condition 3.4 $X^\delta_t$ satisfies (Surpressing the arguments in $p^T(t, r_t)$, $M^T(t, r_t)$ and $\Sigma^T(t, r_t)$):

$$dX^\delta_t = \delta^T_t dp^T + \delta^S_t dp^S$$

$$= \left[ \delta^T T p^T + \delta^S M^S p^S \right] dt + \left[ \delta^T \Sigma^T p^T + \delta^S \Sigma^S p^S \right] dB_t$$  \hspace{1cm} (3.12)

One can set up a Trading Strategy such that the coefficient of the Itô Differential $dB_t$ in equality 3.12 vanishes. This requires:

$$\delta^T \Sigma^T p^T = -\delta^S \Sigma^S p^S$$  \hspace{1cm} (3.13)

This condition makes the Wealth of the portfolio deterministic, thus the portfolio will earn the Short-Rate over the interval $dt$:

$$\frac{dX^\delta_t}{X^\delta_t} := \frac{\delta^T M^T p^T + \delta^S M^S p^S}{X^\delta_t} dt$$

$$= r_t dt$$  \hspace{1cm} (3.14)

One can call the condition in 3.14 a No-Arbitrage condition. Combining 3.11, 3.13 and 3.14 into one expression one finds the general condition that makes sure that deterministic portfolios earn the Short-Rate $r_t$:

$$\frac{dX^\delta_t}{X^\delta_t} = \frac{\delta^T M^T p^T + \delta^S M^S p^S}{X^\delta_t} dt$$

$$= r_t dt \quad \Leftrightarrow$$

$$\frac{\delta^T p^T (M^T - r_t) + \delta^S p^S (M^S - r_t)}{\Sigma^T} = \delta^T \Sigma^T p^T + \delta^S \Sigma^S p^S \quad \Leftrightarrow$$

$$\frac{(M^T - r_t)}{\Sigma^T} = \frac{(M^S - r_t)}{\Sigma^S}$$  \hspace{1cm} (3.15)

The relation in 3.15 holds for all bonds $p^S(t, r_t), \ S \in [0, T]$ and is called the Market Price of Interest Risk:

$$\lambda(t, r_t) = \frac{(M^S - r_t)}{\Sigma^S}, \ t \in [0, T]$$  \hspace{1cm} (3.16)
The Market Price of Interest Risk can also be interpreted with regard to the First Fundamental Theorem of Arbitrage Pricing (cf. Definition 13 on page 22). For this consider the discounted bond process ($C_t$ is the Cash Account cf. page 21):

\[ \tilde{p}_t^T = f(p, c) \]  
\[ = \frac{p^T(t, r_t)}{C_t} \]  

(3.17)  
(3.18)

With Differential Representations for $p^T(t, r_t)$ and $C_t$ (with indexes and arguments suppressed):

\[ dp^T = M^T p^T dt + \Sigma^T p^T dB_t \]  
\[ dC = rC dt \]  

(3.19)  
(3.20)

From the Multi Itô Formula (cf. Definition 10 on page 13) one obtains for $a^{(1)} = M^T p^T$, $a^{(2)} = rC$, $b^{(1)} = \Sigma^T p^T$ and $b^{(2)} = 0$:

\[ d\tilde{p}^T = C^{-1}dp^T - p^T C^{-2}dC + 0 \]
\[ = M^T \tilde{p}^T dt + \Sigma^T \tilde{p}^T dB_t - r \tilde{p}^T dt \]
\[ = (M^T - r) \tilde{p}^T dt + \Sigma^T \tilde{p}^T dB_t \]  

(3.21)

Here the second equality is obtained from inserting the relations 3.19 and 3.20 into the first equality. From the representation in 3.21 one sees that the discounted bond process becomes a martingale under an Equivalent Measure $Q$ if $q$ is replaced by $\lambda(t, r_t)$ in the $Q(\lambda)$-Brownian Motion (cf. Girsanov’s Theorem in Definition 12 on page 18). Thus with:

\[ \tilde{B}_t = B_t + \frac{M^T - r}{\Sigma^T} t, \quad t \in T \]  

(3.22)

The discounted bond price is a martingale under $Q(\lambda)$:

\[ \tilde{p}_t^T = (M^T - r) \tilde{p}^T dt + \Sigma^T \tilde{p}^T (dB_t - \frac{M^T - r}{\Sigma^T} dt) \]
\[ = \Sigma^T \tilde{p}^T dB_t \]  

(3.23)

Hence one sees from the First Fundamental Theorem of Arbitrage that the bond prices are arbitrage free if the the Market Price of Interest has the form of $\lambda(t, r_t)$. In the next section I consider how the Market Price of Interest risk and Arbitrage Pricing Theory can be used to find bond prices.
3.1.3. Bond Pricing

The bond prices in the One Factor Short Rate model of the previous section can be found either:

a By solving the bond Partial Differential Equation under the restriction of $\lambda(t)$ (Market Price of Interest Rate Risk).

b By using so called Affine Bond Price Models.

The Partial Differential Equation that satisfies the No-Arbitrage condition (cf. equality 3.14 on page 24) is obtained by inserting the Drift and Diffusion Coefficients 3.10 & 3.9 from the Bond Partial Differential Equation (cf. equality 3.8) into the Market Price of Interest Risk equation 3.16:

$$bp_2^T(t, r_t)\lambda(t, r_t) = p_1^T(t, r_t) + \frac{1}{2}b^2p_{22}^T(t, r_t) + ap_2^T(t, r_t) - rtp^T(t, r_t)$$

$$p_1^T(t, r_t) + \frac{1}{2}b^2p_{22}^T(t, r_t) + (a - b\lambda(t, r_t))p_2^T(t, r_t) - rtp^T(t, r_t) = 0 \quad (3.24)$$

This Partial Differential Equation with boundary condition $p^T(T, r_T) = 1$ is called the Term Structure Equation. Another way to obtain the Term Structure Equation is to change the Measure from $\mathbb{P}$ to $\mathbb{Q}(\lambda)$ in both the Bond Partial Differential Equation 3.8 and its equivalent Generalized Linear Equation with Multiplicative Noise form from Assumption 2 on page 23 and set their drift terms to zero. Thus under $\mathbb{Q}(\lambda)$ the bond price dynamics are equivalently represented by (Using: $B_t = \tilde{B}_t + \lambda(t, r_t)$) where $\lambda(t, r_t)$ is also given by 3.16:

$$p^T(t, r_t) = p^T(0, r_0) + \int_0^t [p_1^T(s, r_s) + \frac{1}{2}b^2p_{22}^T(s, r_s) + (a - b\lambda(s, r_s)p_2^T(s, r_s))]ds + \int_0^t bp_2^T(s, r_s)d\tilde{B}_s$$

$$p^T(t, r_t) = p^T(0, r_0) + \int_0^t rsp^T(s, r_s)ds + \int_0^t \Sigma^S(s, r_s)p^S(s, r_s)d\tilde{B}_s$$

If one requires the drift terms to be equally zero and drops the integrals one obtains the Term Structure Equation from above:

$$p_1^T(t, r_t) + \frac{1}{2}b^2p_{22}^T(t, r_t) + (a - b\lambda(t, r_t))p_2^T(t, r_t) - rtp^T(t, r_t) = 0$$

The second example makes it explicit that the Term Structure Equation is defined under the Measure $\mathbb{Q}(\lambda)$. It can be solved using the Feynman-Kac Formula which is defined as follows:
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Definition 14. For bounded functions $H(x)$, $\phi(t, x)$, $\mathcal{F}_t = \sigma(B_s, s \leq t)$ and the Itô process $X_t$ with Stochastic Differential Equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t$$

the function

$$f(t, X_t) = E\left[H(X_T)\exp\left(\int_t^T \phi(s, X_s)ds\right)\bigg|\mathcal{F}_t\right], \quad t \in [0, T]$$

solves the Partial Differential Equation

$$df(t, X_t) = \begin{cases} f_1(t, x) + a(t, x)f_2(t, x) + \frac{1}{2}b(t, x)^2f_{22}(t, x) + \phi(t, x)f(t, x) = 0 & \text{if } t < T \\ f(T, x) = H(x) & \text{if } t = T \end{cases}$$

The Feynman-Kac Formula uses the property that a Itô Stochastic Differential Equation becomes a Martingale when its drift term vanishes. From Definition 14 one can see that the process

$$Y_t = f(t, z) = f(t, X_t)\exp\left(\int_0^t \phi(s, X_s)ds\right) = E[f(T, x)\exp\left(\int_0^T \phi(s, X_s)ds\right)|\mathcal{F}_t]$$

(here $Z_t = f(t, X_t)$) is a Martingale. For an Itô Process $X_t$ as in Definition 14 applying the Itô Formula (cf. Definition 9 on page 12) to $Y_t$ and $Z_t$ one obtains (with arguments surpressed):

$$dZ_t = (f_1(t, x) + af_2 + \frac{1}{2}b^2f_{22})dt + bf_2dB_t$$

and

$$dY_t = \phi Y_t dt + \exp\left(\int_0^t \phi ds\right)dZ_t$$

$$= \left(\frac{Y_t}{Z_t}f_1 + \frac{Y_t}{Z_t}af_2 + \frac{1}{2}\frac{Y_t}{Z_t}b^2f_{22} + \phi Y_t dt\right)dt + \frac{Y_t}{Z_t}bf_2dB_t \quad (3.25)$$

Setting the drift term in 3.25 to zero and inserting $Z_t = f(t, X_t)$ one finds the Feynman-Kac Partial Differential Equation for $t < T$:

$$df(t, X_t) = \phi f(t, X_t) + f_1 + af_2 + \frac{1}{2}b^2f_{22} = 0$$

Returning to the bond pricing problem at the beginning of this section one recognizes that the Term Structure Equation in 3.24 can be solved using the Feynman-Kac Formula with $\phi(t, x) = -r_t$ and boundary condition $f(T, x) = P_T(T) = p^T(T, r_t) = 1$. Since the
equation is derived under the Measure $Q(\lambda)$ this leads to the bond price formula for $0 \leq T$:

$$P_t(T) = p_t^T(t, r_t) = E_t^{Q(\lambda)} \left[ P_T(T) \exp \left( - \int_t^T r_s ds \right) | F_t \right], \quad t \in [0, T] \quad (3.26)$$

### 3.1.4. Affine models

A one factor short rate model is **Affine** if the bond prices can be written for deterministic functions $A(t, T)$ and $B(t, T)$:

$$P_t(T) = p_t^T(t, r_t) = \exp \left( A(t, T) - B(t, T) r_t \right), \quad t \in [0, T] \quad (3.27)$$

**Definition 15.** A short rate model is said to provide affine bond prices of the form

$$P_t(T) = p_t^T(t, r_t) = \exp \left( A(t, T) - B(t, T) r_t \right), \quad t \in [0, T]$$

when the smooth, deterministic functions $A(T, T) = B(T, T) = 0$ for $t = T$ and for $0 \leq t < T$ satisfy the Riccati equations

$$\frac{dB(t, T)}{dt} = -m(t) B(t, T) + \frac{1}{2} k(t) B(t, T)^2 - 1$$

$$\frac{dA(t, T)}{dt} = n(t) B(t, T) - \frac{1}{2} l(t) B(t, T)^2$$

if the Short Rate $r_t$ is an Itô process under the Measure $Q(\lambda)$ with

$$r_t = r_0 + \int_0^t a^{Q(\lambda)}(s, r_s) ds + \int_0^t b(s, r_s) d\tilde{B}_s, \quad t \in [0, T]$$

and

$$a^{Q(\lambda)}(t, r_t) = n(t) + m(t) r_t$$

$$b(t, r_t)^2 = l(t) + k(t) r_t$$

for deterministic functions: $m(t), n(t), k(t)$ and $l(t)$.

These necessary conditions can be found by inserting $P_t(T) = p_t^T(t, r_t) = \exp \left( (A(t, T) + B(t, T) r_t) \right)$ into the Term-Structure Equation of 3.24 and replacing $(a - b\lambda(t, r_t))$ by $a^{Q(\lambda)}$ and $b(t, r_t)$ by $b$ in the specification of the Short-Rate in Definition 15. The derivation of the necessary conditions was originally performed in Filipovic (2009,[3]). This yields surpressing arguments in $p_t^T(t, r_t)$.
\[ p_t^T(t, r_t) + \frac{1}{2} b^2 p_{22}^T(t, r_t) + a^{Q(\lambda)} p_2^T(t, r_t) - r_t p_t^T(t, r_t) = 0 \]
\[ \left( \frac{dA(t, T)}{dt} - \frac{dB(t, T)}{dt} r_t \right) p_t^T + \frac{1}{2} b^2 B(t, T)^2 p_t^T - a^{Q(\lambda)} B(t, T)p_t^T - r_t p_t^T = 0 \] (3.28)

For example, to obtain the affine representation of \( a^{Q(\lambda)} \) and \( b(t, r_t)^2 \) one can fix a time point \( t \) and consider Equation: 3.28(after dividing both sides by \( p_t^T \)) for maturities \( T > S > t \).

For invertible Matrix \( M = \begin{pmatrix} B(t, T)^2 & B(t, T) \\ B(t, S)^2 & B(t, S) \end{pmatrix} \):

\[ M \left( \frac{1}{2} b^2 a^{Q(\lambda)} \right) = \begin{pmatrix} \frac{dA(t, T)}{dt} + \left( \frac{dB(t, T)}{dt} - 1 \right) r_t \\ \frac{dA(t, S)}{dt} + \left( \frac{dB(t, S)}{dt} - 1 \right) r_t \end{pmatrix} \]
\[ \left( \frac{1}{2} b^2 a^{Q(\lambda)} \right) = M^{-1} \begin{pmatrix} \frac{dA(t, T)}{dt} + \left( \frac{dB(t, T)}{dt} - 1 \right) r_t \\ \frac{dA(t, S)}{dt} + \left( \frac{dB(t, S)}{dt} - 1 \right) r_t \end{pmatrix} \] (3.29)

Thus \( \left( \frac{b^2}{a^{Q(\lambda)}} \right) \) can be represented as linear functions of \( r_t \):

\[ \left( \frac{b^2}{a^{Q(\lambda)}} \right) \equiv \begin{pmatrix} l(t) + k(t) r_t \\ n(t) + m(t) r_t \end{pmatrix} \]

Moreover, inserting the affine expression for \( \frac{1}{2} b^2 a^{Q(\lambda)} \) into Equation 3.28 yields:

\[ \left( \frac{dA(t, T)}{dt} - \frac{dB(t, T)}{dt} r_t \right) p_t^T + \frac{1}{2} \begin{pmatrix} l(t) + k(t) r_t \\ n(t) + m(t) r_t \end{pmatrix} B(t, T)^2 p_t^T - n(t) + m(t) r_t B(t, T)p_t^T - r_t p_t^T = 0 \]

Equating the terms containing \( r_t \) and those not containing \( r_t \) yields:

\[ \frac{dB(t, T)}{dt} = -m(t) B(t, T) + \frac{1}{2} k(t) B(t, T)^2 - 1 \]
\[ \frac{dA(t, T)}{dt} = n(t) B(t, T) - \frac{1}{2} l(t) B(t, T)^2 \]

The next sections implement the results that were introduced until now in order to find bond prices in the Vasicek and the Cox-Ingersoll-Ross Short Rate models.

### 3.2. Vasicek Model

The Vasicek Model was introduced in 1977 by Oldrich Vasicek and was one of the first approaches to model the short rate through a diffusion process. The intuition of the model is that the Short-Rate follows a mean reverting process. It is constructive to see the discrete analog of the Vasicek model first (For \( \Delta t = 1 \), \( Z_t \sim i.i.d N(0, \sigma^2) \) and constants: \( c, \mu, \sigma \)).
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\[ X_{t+\Delta t} - X_t = c(\mu - X_t)\Delta t + \sigma(B_{t+\Delta t} - B_t) \quad \Leftrightarrow \quad (3.31) \]
\[ X_{t+1} - X_t = c(\mu - X_t) + \sigma(B_{t+1} - B_t) \quad \Leftrightarrow \quad (3.32) \]
\[ X_{t+1} = \alpha + \phi X_t + Z_{t+1} \quad (3.33) \]

The process in equality 3.33 with \( \alpha = c\mu \) and \( \phi = (1 - c) \) is a so called autoregressive process of order 1 (AR(1)). For \( |\phi| \leq 1 \) this Process is stationary and has mean \( \mu \):

\[ X_{t+\Delta t} = \alpha + \phi X_t + Z_{t+\Delta t} \quad \Leftrightarrow \]
\[ (1 - \phi L)X_{t+\Delta t} = \alpha + Z_{t+\Delta t} \quad \Leftrightarrow \]
\[ X_{t+\Delta t} = \frac{\alpha}{1 - \phi} + \frac{Z_{t+\Delta t}}{1 - \phi} \quad \text{with} \]
\[ E(X_{t+\Delta t}) = E(X_t) = \frac{\alpha}{1 - \phi} = \mu + 0 \]

Here the second line is obtained from the first through the use of the Lag-Operator \( (L) \). The Lag-Operator lags the time index of a Random Variable in a sequence of (Time-Indexed) Random Variables: \( L^i X_t = X_{t-i} \). If one lets \( \Delta t \to dt \) and replaces \( X_t \) with \( r_t \) in representation 3.32 then one can formally write:

\[ dr_t = c(\mu - r_t)dt + \sigma dB_t \quad (3.34) \]

with Integral Form:

\[ r_t = r_0 + \int_0^t c(\mu - r_s)ds + \int_0^t \sigma dB_s, \quad t \in [0, T] \quad (3.35) \]

Remembering the representation of the Linear Equation with Additive Noise from 2.30 on page 14 one sees that the Vasicek Stochastic Differential Equation in 3.35 is a special case of the Linear Equation with Additive Noise with: \( c_1(t) = -c, \ c_2(t) = c\mu \) and \( \sigma_2(t) = \sigma \).

The solution for the Vasicek Stochastic Differential Equation in 3.35 is found by using the Solution for the Linear Equation with Additive Noise given by equality 2.36 on page 15. Thus:
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\[ r_t = \exp(-ct) \left( r_0 + \int_0^t \exp(cs)\mu ds + \int_0^t \exp(cs)\sigma dB_s \right) \]

\[ = \exp(-ct)r_0 + \mu[1 - \exp(-ct)] + \exp(-ct) \int_0^t \exp(cs)\sigma dB_s \] (3.36)

In order to price interest rate derivatives the differential representation of the Vasicek Interest Rate is given under the Risk-Neutral Measure \( Q \). One can rewrite the Market Price of Interest Risk for constant bond parameters \( M^T/\Sigma^T = n \) and \( 1/\Sigma^T = m \) as:

\[ \lambda(r_t) = mr_t + n \]

and thus impose an affine interest rate model. Then to change the Measure from \( \mathbb{P} \) to \( Q \) one substitutes \( \tilde{B}_t = B_t + (mr_t + n)t \) into equality 3.34:

\[ dr_t = c(\mu - r_t)dt + \sigma[d\tilde{B}_t - (mr_t + n)dt] \]

\[ = (c\mu + n - (c + \sigma m)r_t)dt + \sigma d\tilde{B}_t \]

\[ = (c + \sigma m) \left( \frac{c\mu + n}{c + \sigma m} - r_t \right) dt + \sigma d\tilde{B}_t \]

\[ := a(b - r_t) + \sigma d\tilde{B}_t \] (3.37)

The solution under \( Q \) of the Differential Representation in 3.37 is found by replacing \( c, \mu \) and \( dB_t \) with \( a, b \) and \( d\tilde{B}_t \) in equation 3.36:

\[ r_t = \exp(-at)r_0 + b[1 - \exp(-at)] + \exp(-at) \int_0^t \exp(as)\sigma d\tilde{B}_s \] (3.38)

Because all parameters are deterministic in 3.38 it follows from the Stochastic Integral that \( r_t \) is a Gaussian process with expectation and variance:

\[ E^Q(r_t) = \mu_r(t) = \exp(-at)r_0 + b[1 - \exp(-at)] \] (3.39)

\[ V^Q(r_t) = q(t) = \sigma^2 \exp(-2at)[\exp(2at) - 1] \]

\[ = \frac{\sigma^2}{2a}[1 - \exp(-2at)] \] (3.40)

The expectation of 3.38 is found by remembering that the expectation of the Itô Stochastic Integral is zero (cf. Definition 5 on page 7). The variance is found by applying the
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Isometry Property given in the same Definition to $\int_0^t \exp (as) \sigma dB_s$. The Vasicek bond price can be obtained in multiple ways, one of them is to use the Feynman-Kac Formula (another is to solve the equations implied by an affine model). Finding the bond price by using the distribution of the short rate process was first outlined in Mamon(2004, [7])

$$P_t(T) = E_{t}^{Q(\lambda)} \left[ \exp \left( - \int_{t}^{T} r_s ds \right) \mid \mathcal{F}_t \right], \quad t \in [0, T]$$

$$= \exp \left( - \int_{t}^{T} E^{Q}(r_s) ds + 0.5 V^{Q} \left( - \int_{t}^{T} r_s ds \right) \right) \quad (3.41)$$

Thus in order to find the bond price one has to integrate the expectation of the Vasicek Solution and find the Variance of the integral of the Vasicek Solution. Integrating the expectation of the Vasicek Solution over the interval $[t, T]$ as given in 3.39:

$$\int_{t}^{T} -E^{Q}(r_s) ds = - \int_{t}^{T} \left( (r_t - b) \exp (-as) + b \right) ds \quad (3.42)$$

$$= \frac{r_t - b}{a} \left[ \exp (-a(T - t)) - 1 \right] - b(T - t) \quad (3.43)$$

$$= -r_t B(t, T) + b B(t, T) - b(T - t) \quad (3.44)$$

Where $B(t, T) := \frac{1 - \exp (-a(T - t))}{a}$ is a deterministic function (Not Brownian Motion). The variance is calculated first for the interval $[0, t]$ for convenience:

$$V^{Q} \left( - \int_{0}^{t} r_s ds \right) = E^{Q} \left[ \left( \int_{0}^{t} r_s ds - \int_{0}^{t} E^{Q}(r_s) ds \right) \left( \int_{0}^{t} r_u du - \int_{0}^{t} E^{Q}(r_u) du \right) \right] \quad (3.45)$$

$$= \int_{0}^{t} \int_{0}^{t} E^{Q}(r_s r_u) ds du - \int_{0}^{t} \int_{0}^{t} E^{Q}(r_s) E^{Q}(r_u) ds du \quad (3.46)$$

$$= \int_{0}^{t} \int_{0}^{t} Cov^{Q}(r_s, r_u) ds du \quad (3.47)$$

Thus before the double integral can be evaluated one needs to find an expression for $Cov^{Q}(r_t, r_s)$ on the interval $[t, s]$:
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\[ \text{Cov}_{Q}(r_t, r_s) = E_{Q} \left( \exp(-at) \int_{0}^{t} \sigma \exp(as) d\tilde{B}_s \exp(-as) \int_{0}^{s} \sigma \exp(au) d\tilde{B}_u \right) \]

\[ = \sigma^2 \exp(-a(t+s)) \int_{0}^{t \wedge s} \exp(2au) du \]

\[ = \frac{\sigma^2}{2a} \exp(-a(t+s))(\exp(2a(t \wedge s)) - 1) \]  

(3.48)

Then inserting \( \text{Cov}_{Q}(r_t, r_s) \) into expression 3.47 and evaluating the two resulting double integrals separately:

\[ \int_{0}^{t} \int_{0}^{t} \frac{\sigma^2}{2a} \exp(-a(u + s))(\exp(2a(u \wedge s)) - 1) ds du = \]

\[ \int_{0}^{t} \int_{0}^{t} \frac{\sigma^2}{2a} \exp(-a(u + s) + 2a(u \wedge s)) ds du - \int_{0}^{t} \int_{0}^{t} \frac{\sigma^2}{2a} \exp(-a(u + s)) ds du \]  

(3.49)

Then expression \[ 1 \] can be evaluated by splitting the inner integral:

\[ \int_{0}^{t} \left[ \int_{0}^{u} \frac{\sigma^2}{2a} \exp(-a(u + s) + 2as) ds + \int_{u}^{t} \frac{\sigma^2}{2a} \exp(-a(u + s) + 2au) ds \right] du = \]

\[ \int_{0}^{t} \frac{\sigma^2}{2a} \left[ 2 - \exp(-au) - \exp(au - at) \right] du \]

\[ = \frac{\sigma^2}{2a^2} \left( 2t + \frac{2}{a} \exp(-at) - \frac{2}{a} \right) \]  

(3.50)

Evaluating expression \[ 2 \] yields:

\[ \int_{0}^{t} \int_{0}^{t} \frac{\sigma^2}{2a} \exp(-a(u + s)) ds du = \]

\[ \int_{0}^{t} \left[ -\frac{\sigma^2}{2a^2} \exp(-au - at) + \frac{\sigma^2}{2a^2} \exp(-au) \right] du = \]

\[ \frac{\sigma^2}{2a^3} (\exp(-2at) - 2 \exp(-at) + 1) \]  

(3.51)

Combining the expression in 3.50 & 3.51 one finds:
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\[ V^Q \left( -\int_0^t r_s ds \right) = 1 - 2 \]

\[ = \frac{\sigma^2}{2a^3} (2ta + 4 \exp(-at) - 3 - \exp(-2at)) \]

\[ = \frac{\sigma^2}{2a^3} 2ta + \frac{\sigma^2}{2a^3} 2(\exp(-at) - 1) + \frac{\sigma^2}{2a^3} (-1 + 2 \exp(-at) - \exp(-2at)) \]

\[ = \frac{\sigma^2}{a^2} t - \frac{\sigma^2}{a^2} B(0, t) - \frac{\sigma^2}{2a} B(0, t)^2 \] (3.53)

Inserting expressions 3.53 (and writing the Variance expression for the interval \([t, T]\)) & 3.44 into the Feynman-Kac formula 3.41 one finds the Vasicek bond price:

\[ P_t(T) = \exp \left( -\int_t^T E^Q(r_s) ds + 0.5V^Q \left( -\int_t^T r_s ds \right) \right) \]

\[ = \exp (-r_t B(t, T) + (\frac{\sigma^2}{2a^2} - b)(T - t) + (b - \frac{\sigma^2}{2a^2}) B(t, T) - \frac{\sigma^2}{4a} B(t, T)^2) \] (3.55)

\[ = \exp (-r_t B(t, T) + A(t, T)) \] (3.56)

Where \( A(t, T) := (\frac{\sigma^2}{2a^2} - b)((T - t) - B(t, T)) - \frac{\sigma^2}{4a} B(t, T)^2 \) is a deterministic function.

Similarly the bond price can be obtained with the equations in Definition 15 because the Vasicek model is affine. The Affine property of a model is useful in the case that it is difficult to find a solution for the Short Rate Stochastic Differential Equation. With the short rate process specified as:

\[ dr_t = a(b - r_t) + \sigma dB_t \]

The task is (for \( m(t) = -a, n(t) = ab, l(t) = \sigma^2, k(t) = 0 \)) to solve the pair of partial differential equations for \( B(T, T) = 0 \) and \( A(T, T) = 0 \):

\[ B(T, T) - B(t, T) = \int_t^T (aB(s, T) - 1) ds \] (3.57)

\[ A(T, T) - A(t, T) = \int_t^T abB(s, T) ds - \frac{1}{2} \int_t^T \sigma^2 B(s, T)^2 ds \] (3.58)

Equation 3.57 implies:

\[ B(t, T) = \int_t^T (-aB(t, s) + 1) ds = \frac{1 - \exp(-a(T - t))}{a} \] (3.59)
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The integral in equation 3.58 yields:

\[
A(t, T) = \frac{\sigma^2}{2} \int_t^T B(t, s)^2 ds - \int_t^T abB(t, s) ds \\
= \frac{\sigma^2}{2a^2} \int_t^T (1 - 2 \exp(-a(s-t)) + \exp(-2a(s-t))) ds + \int_t^T b - b \exp(-a(s-t)) ds \\
= \frac{\sigma^2}{2a^2} \left( (T-t) - \frac{1}{2} \left( \frac{1 - 2 \exp(-a(T-t)) + \exp(-2a(T-t))}{a} - \frac{1 - \exp(-a(T-t))}{a} \right) \\
- b \left( (T-t) - \frac{1 - \exp(-a(T-t))}{a} \right) \right) \\
= \left( \frac{\sigma^2}{2a^2} - b \right) ((T-t) - B(t, T)) - \frac{\sigma^2}{4a} B(t, T)^2
\]

(3.60)

Hence in the Vasicek model bond prices are equivalently obtained through application of the Feynman-Kac equation or the affine model property.

3.2.1. Vasicek calibration

One can calibrate the short rate process by optimizing the Log Likelihood function of the transition densities of the process. For equally spaced observations: \( r_0, r_1, \ldots, r_T \) (with spacing \( \Delta \)) the task becomes to find the parameters that optimize the Log Likelihood function for \( \psi = [a, b, \sigma^2]^{\prime} \):

\[
\arg \max_{a, b, \sigma^2} l_T(\psi) = \sum_{t=1}^T \ln \Phi((r_t; \mu_{r_{t-1}}(\Delta|\psi), q(\Delta|\psi))
\]

(3.61)

Here \( \Delta = t + 1 - t = 1 \) (a one unit increment) and the functions \( \mu_{r_{t-1}}(\Delta|\psi) \) and \( q(\Delta|\psi) \) are given in equations 3.39 & 3.40 and \( \Phi(r; m(), v()) = \frac{1}{\sqrt{2\pi v()}} \exp \left( -\frac{(r-m())^2}{2v()} \right) \) is the normal density.

As starting values one can use the parameter estimates of the linear regression:

\[
r_{t+\Delta t} - r_t = a(b - r_t) + \sigma Z_{\Delta t}
\]

The calibration results for the data in figure 3.1 are given as:
Table 3.1.: Log Likelihood Calibration of the Vasicek Short Rate Model

A check for the stability of the parameters can be done by reestimating the starting values for the calibration over a observation period. Here the parameters are reestimated with fixed start date: Jan 2004 and increasing end date: Jan 2005 + Day\textsubscript{i}, \( i \in \{1, 2, \ldots, 252\} \). At \( i = 252 \) the full sample is estimated.

It seems that the starting values are not stable, which raises the question whether the 1w Euribor interest rate is correctly specified by the Vasicek model. One disadvantage of the Vasicek specification is that the Short Rate can take negative values. The Cox-Ingersoll-Ross specification which will be considered in the next section is an example of a Short Rate model that does not have this problem.
3.3. Cox-Ingersoll-Ross Model

The Cox-Ingersoll-Ross (CIR) Model was introduced in 1985 and specifies that the Short Rate process is mean reverting but never negative. The Stochastic Differential Equation of the model under the risk neutral measure $\mathbb{Q}$ reads as:

$$r_t = r_0 + \int_0^t a(b - r_s)ds + \sigma \sqrt{r_t} \sigma \int_0^t dB_s \quad t \in [0, T] \tag{3.62}$$

with Differential Form:

$$dr_t = a(b - r_t)dt + \sigma \sqrt{r_t} d\tilde{B}_t \tag{3.63}$$

Unlike the Vasicek model the Short Rate $r_t$ is not a Gaussian Process in the Cox-Ingersoll-Ross model. Instead the conditional Short Rate is distributed as a non-central Chi-Square Random Variable. The dynamics in equation 3.63 can be derived if $r_t$ is defined as:

$$r_t := \sum_{i=1}^d X_{i,t}^2 = X_{1,t}^2 + X_{2,t}^2 + \cdots + X_{d,t}^2, \quad d \geq 1$$

where the process $X_{i,t}, \ i \in \{1, \ldots, d\}$ is specified as a Linear Equation with Additive Noise:

$$X_{i,t} - X_{i,0} = \int_0^t -\frac{1}{2} a X_{i,s} ds + \int_0^t \frac{1}{2} \sigma dB_{i,s}$$

Applying the Multi Itô Formula (cf. Definition 10 on page 13) to $r_t = f(x_1^2 + x_2^2 + \cdots + x_d^2)$ yields the partial derivatives for $i \in [1, d]$:

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial x_i} = 2x_i$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} 2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and replacing $x_i^2$ by $X_{i,t}^2$: 

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\[ f(\sum_{i=1}^{d} X_{i,t}^2) - f(\sum_{i=1}^{d} X_{i,0}^2) = \sum_{i=1}^{d} \int_{0}^{t} 2X_{i,s}dX_{i,s} + 0.5 \sum_{i=1}^{d} \int_{0}^{t} \frac{2}{4} \sigma^2 ds \]
\[ = \int_{0}^{t} \frac{d}{4} \sigma^2 ds - \sum_{i=1}^{d} \int_{0}^{t} aX_{i,s}^2 ds + \sum_{i=1}^{d} \int_{0}^{t} \sigma X_{i,s} dB_{i,s} \]
\[ = \int_{0}^{t} (\frac{d}{4} \sigma^2 - ar_s)ds + \sigma \sqrt{r_t} \sum_{i=1}^{d} \int_{0}^{t} \frac{X_{i,s}}{\sqrt{s}} dB_{i,s} \]
\[ = \int_{0}^{t} (\frac{d}{4} \sigma^2 - ar_s)ds + \sigma \sqrt{r_t} \int_{0}^{t} dB_s \]

The last equality follows because the increments of Brownian Motion are uncorrelated \((E(dB_t, dB_s) = 0, \text{ for } t \neq s)\). Thus with \(d = ah\frac{4}{\sigma^2}\) one obtains the Cox-Ingersoll-Ross specification as in 3.62. And since the process \(X_{i,t}\) is Gaussian one can deduce that the process \(r_t = \sum_{i=1}^{d} X_{i,T}^2\) is noncentral Chi-Square distributed.

The expectation of the short rate is found by taking the expectation (conditional on \(r_0\)) of equation 3.62 and then solving the resulting ODE:

\[ E(r_t) := \mu_r(t) = \mu_r(0) + \int_{0}^{t} a(b - \mu_r(s))ds \]
\[ \mu_r(t)' = a(b - \mu_r(t)) \]
\[ \int_{\mu_r(0)}^{\mu_r(t)} \frac{d\mu_r(s)}{b - \mu_r(s)} = a \int_{0}^{t} ds \]
\[ \ln (b - \mu_r(t)) - \ln (b - \mu_r(0)) = -at \]
\[ \mu_r(t) = r_0 \exp (-at) + b(1 - \exp (-at)) \] (3.64)

The last line follows by writing \(\mu_r(0) = r_0\). The variance is found using the identity \(\text{Var}(r_t) = E(r_t^2) - E(r_t)^2\) after solving the ODE of \(E(r_t^2)\):

\[ E(r_t^2) := q_r(t) = q_r(0) + \int_{0}^{t} \mu_r(s)(\sigma^2 + 2ab)ds - 2a \int_{0}^{t} q_r(s) ds \]
\[ q(t)' = \mu_r(t)(\sigma^2 + 2ab) - 2aq(t) \] (3.65)

Equation 3.65 is a Non-Seperable First-Order Linear Differential Equation that can be solved using the Integrating Factor \(\mu = \exp (\int_{0}^{t} 2ads) = \exp (2at)\):
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\[ q(t) = \mu^{-1} \int_{0}^{t} (r_{0} \exp(rs) + b \exp(2as) - \exp(as)) (\sigma^{2} + 2ab) ds + \mu^{-1} q(0) \]

\[ = \mu^{-1} \left[ \left\{ r_{0} \left( \frac{1}{a} \exp(at) - \frac{1}{a} \right) + b \left( \frac{1}{2a} \exp(2at) - \frac{1}{a} \exp(at) + \frac{1}{2a} \right) \right\} (\sigma^{2} + 2ab) \right] + \mu^{-1} q(0) \]

\[ = \left[ \frac{r_{0}}{a} (\exp(-at) - \exp(-2at)) + \frac{b}{2a} (1 - 2 \exp(-at) + \exp(-2at)) \right] (\sigma^{2} + 2ab) \]

\[ + \exp(-2at)q(0) \quad (3.66) \]

At the same time \( E(r_{t})^{2} = \mu_{r}(t)^{2} \) is given by:

\[ \mu_{r}(t)^{2} = q(0) \exp(-2at) + 2r_{0}b(\exp(-as) - \exp(-2as)) + b^{2}(1 - \exp(-as))^{2} \] (3.67)

Subtracting equation 3.67 from 3.66 yields the Variance of the Short rate:

\[ Var(r_{t}) = q(t) - \mu_{r}(t)^{2} = \left( \frac{r_{0}}{a} (\sigma^{2} + 2ab) - 2r_{0}b \right) (\exp(-at) - \exp(-2at)) \]

\[ + \left( \frac{b}{2a} (\sigma^{2} + 2ab) - b^{2} \right) (1 - \exp(-at))^{2} \]

\[ = \frac{r_{0}\sigma^{2}}{a} (\exp(-at) - \exp(-2at)) + \frac{b\sigma^{2}}{2a} (1 - \exp(-at))^{2} \] (3.68)

Thus the first two moments in the CIR Short Rate specification (Using the superscript C for CIR ) are given as:

\[ E(r_{t}) = \mu^{C}_{r}(t) = r_{0} \exp(-at) + b(1 - \exp(-at)) \] (3.69)

\[ Var(r_{t}) = q^{C}_{r}(t) = \frac{r_{0}\sigma^{2}}{a} (\exp(-at) - \exp(-2at)) + \frac{b\sigma^{2}}{2a} (1 - \exp(-at))^{2} \] (3.70)

The bond prices are calculated in this affine model by solving the ordinary differential equations for \( A(t,T) \) and \( B(t,T) \) from Definition 15 on page 28 with \( m(t) = -a \), \( n(t) = ab \), \( \sigma^{2} = k(t) \) and \( l(t) = 0 \).

The Solution for the equation of \( B(t,T) \) can be found by factoring the ordinary differential equation (surpressing the arguments):

\[ \frac{dB}{dt} = aB^{2} - bB - 1 = (B - c_{1})(B - c_{2}) \]
where \( c_1 \) and \( c_2 \) are:

\[
\begin{align*}
    c_1 &= -a + \frac{\sqrt{a^2 + 2\sigma^2}}{\sigma^2} \\
    c_2 &= -\frac{2}{-a + \sqrt{a^2 + 2\sigma^2}} 
\end{align*}
\]

Integrating both sides of the differential equation and evaluating results in (Using:
\[
\int \frac{2}{\sigma^2} dB(s, T) - c_2) = \frac{\ln(B - c_2)}{c_1 - c_2} + c
\]

\[
\int_{B(t, T)}^{B(T, T)} \frac{2}{\sigma^2} dB(s, T) = \int_t^T ds \
\Rightarrow \frac{2}{\sigma^2} \left[ \ln(B - c_2) - \ln(B(s, T) - c_2) \right] = T - t \\
- \ln(B(t, T) - c_1)(c_1 - c_2) - \ln(B(t, T) - c_2)(c_2 - c_1) ) = \\
- \frac{\sigma^2}{2} (T - t)(c_2 - c_1)(c_1 - c_2) - \ln(c_1 - c_2)(c_1 - c_2) \\
\Rightarrow
\]

\[
\begin{align*}
    B(t, T) &= \frac{c_2}{c_1} \exp \left\{ -\frac{\sigma^2}{2} (T - t)(c_2 - c_1) \right\} \\
    B(t, T) &= \frac{c_2(1 - \exp \{ (T - t)(c_2 - c_1)\})}{1 - \exp \left\{ -\frac{\sigma^2}{2} (T - t)(c_2 - c_1) \right\}} 
\end{align*}
\]

(3.71)

Substituting into the numerator of equation 3.71 the expression for \( c_2 \) and substituting into numerator and denominator \( c_2 - c_1 = -\frac{2}{\sigma^2} \sqrt{a^2 + 2\sigma^2} := -\frac{2}{\sigma^2} \gamma \) the expression for \( B(t, T) \) becomes:

\[
\begin{align*}
    B(t, T) &= \frac{-2(1 - \exp -\frac{\sigma^2}{2} \{ (T - t)(c_2 - c_1)\})}{c_1 \sigma^2 - \sigma^2 \exp -\frac{\sigma^2}{2} \{ (T - t)(c_2 - c_1)\} c_2} \\
    &= \frac{2(\exp (\gamma(T - t)) - 1)}{(\gamma + a)(\exp (\gamma(T - t)) - 1) + 2\gamma} 
\end{align*}
\]

(3.72)

The form of the deterministic function \( A(t, T) \) is found by solving the partial differential equation:
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\[
\frac{dA(t, T)}{dt} = abB(t, T) \quad \Leftrightarrow \quad \int_{A(t, T)}^{A(T, T)} dA(s, T) = \int_{t}^{T} abB(s, T) \, ds \quad \Leftrightarrow \quad (3.74) \\
A(T, T) - A(t, T) = \int_{t}^{T} abB(s, T) \, ds \quad (3.75)
\]

Evaluating the RHS integral in 3.74 at the upper bound and inserting the expression for \( B(t, T) \) yields:

\[
A(t, T) = \int_{t}^{T} -ab \frac{2(\exp \gamma(s - t) - 1)}{(\gamma + a)(\exp (\gamma(s - t) - 1) + 2\gamma)} \, ds \quad (3.76)
\]

This integral can be evaluated by splitting it:

\[
A(t, T) = -2ab \int_{t}^{T} \underbrace{\frac{\exp (\gamma(s - t))}{(\gamma + a)(\exp (\gamma(s - t)) - 1) + 2\gamma}}_{1} \, ds \\
+ 2ab \int_{t}^{T} \underbrace{\frac{1}{(\gamma + a)(\exp (\gamma(s - t)) - 1) + 2\gamma}}_{2} \, ds \quad (3.77)
\]

Evaluating the first expression results in:

\[
[1] = -\frac{2ab}{\gamma(\gamma + a)} \left[ \ln \left( \{\gamma + a\}\{\exp (\gamma(T - t)) - 1\} \right) - \ln (2\gamma) \right] \\
= -\frac{2ab}{\gamma(\gamma + a)} \left[ \ln \left( \frac{(\gamma + a)(\exp (\gamma(T - t)) - 1) + 2\gamma}{2\gamma} \right) \right] \quad (3.78)
\]

The second expression becomes:

\[
[2] = +2ab \left[ \frac{(T - t)}{\gamma - a} - \frac{1}{\gamma(\gamma - a)} \{\ln ((\gamma + a)(\exp (\gamma(T - t)) - 1) + 2\gamma)\} - \ln (2\gamma) \right] \\
= \frac{2ab(T - t)}{\gamma - a} - \frac{-2ab}{\gamma(\gamma - a)} \left[ \ln \left( \frac{(\gamma + a)(\exp (\gamma(T - t)) - 1) + 2\gamma}{2\gamma} \right) \right] \quad (3.79)
\]
Combining the expressions in 3.78 and 3.79 and substituting in the expression for $\gamma$ yields:

$$A(t, T) = \left( \frac{-2ab}{\gamma(\gamma + a)} - \frac{2ab}{\gamma(\gamma - a)} \right) \ln \left( \frac{2\gamma}{(\gamma + a)(\exp(\gamma(T - t)) - 1) + 2\gamma} \right) + \frac{2ab(T - t)}{\gamma - a}$$

$$= \ln \left( \frac{2\gamma \exp \left( 0.5(T - t)(\gamma + a) \right)}{(\gamma + a)(\exp(\gamma(T - t)) - 1) + 2\gamma} \right)^{\frac{2ab}{\sigma^2}}$$

With the functions $A(t, T)$ and $B(t, T)$ bond prices can be calculated in the CIR model. Similarly, as in the section of the Vasicek Short Rate model one can calibrate the Short Rate model to the data. But unlike the Vasicek model the transition densities in the Cox-Ingersoll-Ross model are Chi-Square and not Normally distributed.

### 3.3.1. CIR calibration

The following calibration follows the steps outlined by Kamil Kladivko ([6]). For the calibration the 1w EURIBOR for about 11 years is taken.

The CIR process is well defined if $a, b, \sigma$ are all positive and

$$2ab \geq \sigma^2$$

The true transition density of the CIR process is given by:

$$p(r_t | r_{t-\Delta t}; \psi, \Delta t) = c \exp \left( -u - v \right) \left( \frac{v}{a} \right)^{\frac{q}{2}} I_q \left( 2\sqrt{uv} \right)$$

$$c = \frac{2a}{\sigma^2 \left( 1 - \exp(-a\Delta t) \right)}$$

$$u = cr_{t-\Delta t} \exp \left( -a\Delta t \right)$$

$$v = cr_t$$

$$q = \frac{2ab}{\sigma^2} - 1$$

Here $I_q(2\sqrt{uv})$ is the modified Bessel function of the first kind of order $q$. For integer values of $q$ $I_q(x)$ admits representation:

$$I_q(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + q + 1)} \left( \frac{x}{2} \right)^{2m+q}$$
Thus, the maximum likelihood estimation problem for parameter vector \( \psi = [a, b, \sigma^2]' \) is given by:

\[
\arg \max_{a,b,\sigma^2} l_T(\psi) = \\
T \sum_{t=1}^T \ln (c) - (cr_{t-1} \exp (-a\Delta t) + cr_t) + \frac{q}{2}(\ln (cr_t) - \ln (cr_{t-1} \exp (-a\Delta t))) \\
+ \ln \left( I_q(2\sqrt{uv}) \right)
\]  

(3.85)

For implementation reasons often the scaled version \( I_q(x)^{sv} = \exp(-x)I_q(x) \) of the modified bessel function is used. This implies the changed maximum likelihood problem:

\[
\arg \max_{a,b,\sigma^2} l(\psi) = \\
T \sum_{t=1}^T \ln (c) - (cr_{t-1} \exp (-a\Delta t) + cr_t) + \frac{q}{2}(\ln (cr_t) - \ln (cr_{t-1} \exp (-a\Delta t))) \\
+ \ln \left( I_q^{sv}(2\sqrt{uv}) \right) + 2\sqrt{uv}
\]  

(3.86)

In the following \( \Delta t := 1 \). As starting values for the optimization problem in eq: 3.85 one can use the parameter estimates of the linear regression:

\[
\frac{r_{t+\Delta t} - r_t}{\sqrt{r_t}} = a(b - r_t) + \sqrt{r_t} \sigma Z_{\Delta t} \\
\iff \\
\frac{r_{t+\Delta t} - r_t}{\sqrt{r_t}} = ab - a \sqrt{r_t} + \sigma Z_{\Delta t}
\]  

(3.87)

Here \( Z_{\Delta t} \) denotes a \( N(0, \Delta t) \) random variable. The calibration to the full data yields

<table>
<thead>
<tr>
<th>Regression</th>
<th>Max Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.0980</td>
</tr>
<tr>
<td>b</td>
<td>2.0898</td>
</tr>
<tr>
<td>sig</td>
<td>0.0116</td>
</tr>
<tr>
<td>L</td>
<td>-1346.395</td>
</tr>
</tbody>
</table>

Table 3.2.: LogLikelihood Calibration of Cox-Ingersoll-Ross Short Rate Model

The check for parameter stability as performed in the Vasicek section yields:
Chapter 3. Modeling

Figure 3.3.: Reestimated parameters $a$, $b$, $\sigma$. Sample start: 01/2004, Reestimation start: 01/2005, Sample end: 01/2006

This plot also suggests that the CIR interest rate model does not appropriately specify the 1w Euribor interest rate.
Chapter 4.

Conclusion

In this thesis the theoretical framework for modeling zero coupon bond prices in the affine interest rate models Vasicek and Cox-Ingersoll-Ross were presented. The prices of the implied zero coupon bonds in the Vasicek model were derived using the distribution of the short rate and equivalently by solving the equations induced by an affine model. The CIR zero coupon bond prices were derived solving the equations of an affine model only. Practically, the models were calibrated to the 1w Euribor interest rate for the period of: 01/2004-01/2006 with results that suggest that both interest rates are not capable of capturing the dynamics of the data.
Appendix A.

R Code

```r
# Vasicek Initial Values Regression
# NOTE: euribor is the DataSet
#

deltaT = 1;
x <- euribor;
x1 <- lx <- x[1:(length(x)−1)];
dx <- diff(x);

dataframe <- data.frame(dx, x1);
summary(reg <- lm(dx ~ ., dataframe));
start.a <- reg$coefficients[2];
start.b <- reg$coefficients[1]/start.a;
start.sig <- sd(reg$residuals)/sqrt(deltaT);
start.coefs <- c(start.a, start.b, start.sig);

# Vasicek Optimization
#

VasicekObjective <- function(Param, Data) {
  data_ <- Data;
  obs_ <- length(Data);

  dataForward <- data_[2:obs_];
dataLag <- data_[1:(obs_-1)];
timeStep_ <- deltaT;
a_ <- Param[1];
b_ <- Param[2];
sig_ <- Param[3];

  mu_ <- exp(-a_*timeStep_)*dataLag + b_*(1-exp(-a_*timeStep_))
  v_ <- sig_~2*(1-exp(-2*a_*timeStep_))/(2*a_);
}
```

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Appendix A. R Code

\[
\text{lnL} \leftarrow -\sum \log (\text{dnorm}(\text{dataForward}, \mu, \sqrt{\text{v}})) / \text{obs} ;
\]
\[
\text{return} (\text{lnL}) ;
\]
\[
\text{nlm.vasicek} \leftarrow \text{nlm}(\text{VasicekObjective}, p=\text{start.coeffs}, \text{Data}=\text{x}, \text{iterlim} = 250, \text{hessian}=\text{TRUE}) ;
\]

### Vaseick Iterated Parameter Calculation

\[
\text{End} \leftarrow \text{length(x)/2} ;
\]
\[
\text{ts.start.coeffs} \leftarrow 0 ;
\]
\[
\text{for} \ (i \ in \ 1:\text{End}) 
\]
\[
\{ 
\text{x1} \leftarrow \text{l x} \leftarrow \text{x}[1:((\text{End}+i)-1)]; 
\text{x2} \leftarrow \text{nx} \leftarrow \text{x}[2:((\text{End}+i)] ;
\text{dx} \leftarrow \text{nx} - \text{l x} ;

\text{dataframe} \leftarrow \text{data.frame(dx,x1)} ;
\text{summary}(\text{reg} \leftarrow \text{lm(dx} \sim ., \text{data}=\text{dataframe})) ;
\text{start.a} \leftarrow -\text{reg$coefficients[2]} ;
\text{start.b} \leftarrow \text{reg$coefficients[1]/start.a} ;
\text{start.sig} \leftarrow \text{sd(\text{reg$residuals})/sqrt(\text{deltaT})} ;
\text{start.coeffs} \leftarrow \text{c(start.a, start.b, start.sig)} ;
\text{ts.start.coeffs} \leftarrow \text{rbind(ts.start.coeffs, start.coeffs)} ;
\}
\text{ts.start.coeffs} \leftarrow \text{ts.start.coeffs}[-1,] ;
\]

### CIR Initial Values Regression

\[
\text{deltaT} = 1 ;
\text{x} \leftarrow \text{euribor} ;
\text{l x} \leftarrow \text{x}[1:((\text{length(x)}-1)] 
\text{x1} \leftarrow \text{deltaT}/\text{sqrt(l x)} ;
\text{x2} \leftarrow \text{deltaT}\ast\text{sqrt(l x)} ;
\text{dx} \leftarrow \text{diff(x)/sqrt(l x)} ;
\text{dataframe} \leftarrow \text{data.frame(dx,x1,x2)} ;
\text{summary}(\text{reg} \leftarrow \text{lm(dx} \sim 0 + ., \text{dataframe})) ;
\text{start.a} \leftarrow -\text{reg$coefficients[2]} ;
\text{start.b} \leftarrow \text{reg$coefficients[1]/start.a} ;
\text{start.sig} \leftarrow \text{sd(\text{reg$residuals})/sqrt(\text{deltaT})} ;
\text{start.coeffs} \leftarrow \text{c(start.a, start.b, start.sig)} ;
\]
Appendix A. R Code

# CIR Optimization

CIRobjective <- function(Param, Data) {
    data_ <- Data;
dataForward_ <- data_[2:length(Data)];
dataLag_ <- data_[1:(length(Data)-1)];
obs_ <- length(Data);
timeStep_ <- deltaTime;
a_ <- Param[1];
b_ <- Param[2];
sig_ <- Param[3];
c_ <- 2*a_/(sig_^2*(1-exp(-a_*timeStep_)));
q_ <- 2*a_*b_/(sig_^2-1);
u_ <- c_*exp(-a_*timeStep_)*dataLag_;
v_ <- c_*dataForward_;
z_ <- 2*sqrt(u_ *v_);
bf_ <- besselI(z_,q_,expon.scaled=TRUE);
lnL <- -(obs_-1)*log(c_) + sum(u_+v_ -0.5*q_*log(v_/u_)) -
       log(bf_) - z_;
return(lnL);
}

# CIR Iterated Parameter Calculation

End <- length(x)/2;
ts.start.coeffs <- 0;
for (i in 1:End) {
    lx <- x[1:(End+i)-1];
x <- x[2:(End+i)];
x1 <- deltaTime/sqrt(lx);
x2 <- deltaTime*sqrt(lx);
dx <- (nx-lx)/sqrt(lx);
dataframe <- data.frame(dx,x1,x2);
summary(reg <- lm(dx ~ 0 + .,data=dataframe));
start.a <- -reg$coefficients[2];
start.b <- reg$coefficients[1]/start.a;
start.sig <- sd(reg$residuals)/sqrt(deltaTime);
start.coeffs <- c(start.a,start.b,start.sig);
ts.start.coeffs <- rbind(ts.start.coeffs,start.coeffs)
}
ts.start.coeffs <- ts.start.coeffs[-1,]
Bibliography


